PHYSICS 523, QUANTUM FIELD THEORY II
Homework 3
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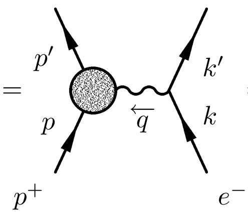
JACOB LEWIS BOURJAILY

The Rosenbluth Formula

We are to prove the *Rosenbluth Formula* by considering the elastic scattering of a relativistic electron off of a proton while correcting the vertex function of the proton. The amplitude for this process is,

$i\mathscr{M} =$  $= \overline{u}(k')(-ie\gamma_\mu)u(k)\frac{-i}{q^2}\overline{u}(p')(-ie\Gamma^\mu)u(p).$

a) Let us simplify the amplitude using the Gordon identity. Recall that we showed in class that the generalized vertex function Γ^μ may be written in terms of functions $F_1(q^2)$ and $F_2(q^2)$ as

$$\Gamma^\mu = \gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2m}F_2(q^2).$$

Inserting this into the amplitude and recalling the Gordon identity, we see that

$$\begin{aligned}
i\mathscr{M} &= i\frac{e^2}{q^2}\overline{u}(k')\gamma_\mu u(k)\overline{u}(p')\Gamma^\mu u(p),\\
&= i\frac{e^2}{q^2}\overline{u}(k')\gamma_\mu u(k)\overline{u}(p')\left(\gamma^\mu F_1 + \frac{i\sigma^{\mu\nu}q_\nu}{2m}F_2\right)u(p),\\
&= i\frac{e^2}{q^2}\overline{u}(k')\gamma_\mu u(k)\overline{u}(p')\left(\gamma^\mu F_1 + \frac{i\sigma^{\mu\nu}q_\nu}{2m}F_2 + \frac{(p'+p)^\mu}{2m}F_2 - \frac{(p'+p)^\mu}{2m}F_2\right)u(p),\\
&= i\frac{e^2}{q^2}\overline{u}(k')\gamma_\mu u(k)\overline{u}(p')\left(\gamma^\mu(F_1+F_2) - \frac{(p'+p)^\mu}{2m}F_2\right)u(p),
\end{aligned}$$

$$\therefore \Gamma^\mu = \gamma^\mu(F_1+F_2) - \frac{(p'+p)^\mu}{2m}F_2.$$

b) Let us compute the spin-averaged amplitude squared directly. We see that

$$\begin{aligned}
\overline{|\mathscr{M}|^2} &= \frac{e^4}{4q^4}\sum_{\text{spin}}\overline{u}(k')\gamma_\mu u(k)\overline{u}(p')\left(\gamma^\mu(F_1+F_2) - \frac{(p'+p)^\mu}{2m}F_2\right)u(p)\overline{u}(p)\left(\gamma^\nu(F_1+F_2) - \frac{(p'+p)^\nu}{2m}F_2\right)u(p')\overline{u}(k)\gamma_\nu u(k'),\\
&= \frac{e^4}{4q^4}\text{Tr}\left[(\not{k}'+m_e)\gamma_\mu(\not{k}+m_e)\gamma_\nu\right]\times\\
&\quad\left\{(F_1+F_2)^2\text{Tr}\left[(\not{p}'+m)\gamma^\mu(\not{p}+m)\gamma^\nu\right] - F_2(F_1+F_2)\frac{(p'+p)^\nu}{2m}\text{Tr}\left[(\not{p}'+m)\gamma^\mu(\not{p}+m)\right]\right.\\
&\quad\left.-F_2(F_1+F_2)\frac{(p'+p)^\mu}{2m}\text{Tr}\left[(\not{p}'+m)(\not{p}+m)\gamma^\nu\right] + F_2^2\frac{(p'+p)^\mu(p'+p)^\nu}{4m^2}\text{Tr}\left[(\not{p}'+m)(\not{p}+m)\right]\right\},\\
&= \frac{4e^4}{q^4}\left(k'_\mu k_\nu + k'_\nu k_\mu - g_{\mu\nu}(k'\cdot k - m_e^2)\right)\times\\
&\quad\left\{(F_1+F_2)^2\left(p'^\mu p^\nu + p'^\nu p^\mu - g^{\mu\nu}(p'\cdot p - m^2)\right) - F_2(F_1+F_2)(p'+p)^\mu(p'+p)^\nu\right.\\
&\quad\left.+\frac{F_2^2}{4m^2}(p'+p)^\mu(p'+p)^\nu(p'\cdot p + m^2)\right\},\\
&= \frac{4e^4}{q^4}\left(k'_\mu k_\nu + k'_\nu k_\mu - g_{\mu\nu}(k'\cdot k - m_e^2)\right)\times\\
&\quad\left\{(F_1+F_2)^2\left(p'^\mu p^\nu + p'^\nu p^\mu - g^{\mu\nu}(p'\cdot p - m^2)\right) + (p'+p)^\mu(p'+p)^\nu\left(\frac{p'\cdot p + m^2}{4m^2}F_2^2 - F_2(F_1+F_2)\right)\right\},
\end{aligned}$$

$$\begin{aligned}
\therefore \overline{|\mathscr{M}|^2} = \frac{8e^4}{q^4}\Bigg[&(F_1+F_2)^2\left(k'\cdot p'k\cdot p + k'\cdot pk\cdot p - k'\cdot km^2 - p'\cdot pm_e^2 + 2m^2m_e^2\right)\\
&+\left(\frac{p'\cdot p + m^2}{4m^2}F_2^2 - F_2(F_1+F_2)\right)\left(k'\cdot(p'+p)k\cdot(p'+p) - \frac{1}{2}(k'\cdot k - m_e^2)(p'+p)^2\right)\Bigg].
\end{aligned}$$

c) Let us consider the kinematics of this reaction in the initial rest frame of the proton. In this frame we see that $p = (m, \vec{0}), k = (E, E\hat{z}), k' = (E', \vec{k'}), p' = (E - E' + m, -\vec{k})$ with $|\vec{k'}| = E'$. We have defined the momentum transfer q such that $p' - p = q = k - k'$.

Noting that $p \cdot p' = m^2 + Em - E'm$, let us compute p'^2.

$$p'^2 = (p+q)^2 = p^2 + 2p\cdot q + q^2 = m^2 + 2p\cdot(p'-p) + q^2 = -m^2 + 2p'\cdot p + q^2 = m^2 + 2Em - 2E'm + q^2 = m^2,$$
$$\implies q^2 = 2E'm - 2Em,$$
$$\therefore E' = E + \frac{q^2}{2m}.$$

If we write $k' = (E', 0, E'\sin\theta, \cos\theta)$ so that $q = (E - E', 0, -E'\sin\theta, E - E'\cos\theta)$ we see

$$q^2 = E'^2 - 2EE' + E^2 - E'^2 - E'^2\sin^2\theta - E^2 + 2EE'\cos\theta - E'^2\cos^2\theta = 2EE'(\cos\theta - 1) = -4EE'\sin^2\frac{\theta}{2}.$$

Using our identity derived above that $E' = E + \frac{q^2}{2m}$, we may conclude that

$$q^2 = -4E^2\sin^2\frac{\theta}{2} - \frac{q^2}{2m}4E\sin^2\frac{\theta}{2},$$
$$\therefore q^2 = -\frac{4E^2\sin^2\frac{\theta}{2}}{1 + \frac{2E}{m}\sin^2\frac{\theta}{2}}.$$

Let us now compute all of the required inner products to compute the desired amplitude squared. Noting that $p^2 = p'^2 = m^2, k^2 = k'^2 = 0$, and $p \cdot k = Em$ we may derive all of our necessary identities and inner products indirectly (it's more fun that way). We notice that

$$p'^2 = m^2 = p^2 + 2p\cdot q + q^2 = -m^2 + 2p'\cdot p + q^2,$$
$$\therefore p'\cdot p = m^2 - \frac{q^2}{2}.$$

Similarly,

$$p'^2 = m^2 = p^2 + 2p\cdot q + q^2 = m^2 + 2p\cdot k - 2p\cdot k' + q^2,$$

but we know that $p\cdot k = Em$,

$$\therefore p\cdot k' = Em + \frac{q^2}{2}.$$

Likewise,

$$k'^2 = 0 = k^2 - 2k\cdot q + q^2 = 2k\cdot k' + q^2 = 0,$$
$$\therefore k'\cdot k = -\frac{q^2}{2}.$$

And

$$k'^2 = 0 = k^2 - 2k\cdot q + q^2 = -2k\cdot p' + 2k\cdot p + q^2,$$

where we know that $k\cdot p = Em$ and

$$\therefore k\cdot p' = Em + \frac{q^2}{2}.$$

Similarly,

$$k^2 = 0 = k'^2 + 2q\cdot k' + q^2 = 2p'\cdot k' + q^2,$$
$$\therefore p'\cdot k' = Em.$$

Tabulating our results, we have shown that

$$k'\cdot k = -\frac{q^2}{2} \qquad p'\cdot p = m^2 - \frac{q^2}{2} \qquad k'\cdot p = Em + \frac{q^2}{2}$$
$$p'\cdot k = Em + \frac{q^2}{2} \qquad k'\cdot p' = Em \qquad p\cdot k = Em.$$

These imply that

$$k\cdot(p'+p) = 2Em + \frac{q^2}{2}, \quad k'\cdot(p'+p) = 2Em + \frac{q^2}{2}, \quad \text{and} \quad (p+p')^2 = 4m^2 - q^2.$$

d) We are to use the kinematic information derived in part (c) above to rewrite the spin-averaged amplitude squared into a more convenient form. Recall that

$$\overline{|\mathscr{M}|^2} = \frac{8e^4}{q^4}\Bigg[(F_1+F_2)^2 \overbrace{\left(k'\cdot p' k\cdot p + k'\cdot p k\cdot p - k'\cdot k m^2 - p'\cdot p m_e^2 + 2m^2 m_e^2\right)}^{\text{i}} + \underbrace{\left(\frac{p'\cdot p + m^2}{4m^2}F_2^2 - F_2(F_1+F_2)\right)}_{\text{ii}}\underbrace{\left(k'\cdot(p'+p)k\cdot(p'+p) - \frac{1}{2}(k'\cdot k - m_e^2)(p'+p)^2\right)}_{\text{iii}}\Bigg].$$

We note that in the approximation where $k^2 \sim 0$, we should set $m_e \to 0$. Let us compute each part separately first before combining the results.

i. $$(k'\cdot p')(k\cdot p) + (k'\cdot p)(k\cdot p') - (k'\cdot k)m^2 = (Em)^2 + (Em)^2 + Emq^2 + \frac{q^4}{4} + \frac{q^2}{2}m^2.$$

ii.
$$\begin{aligned}\frac{p'\cdot p + m^2}{4m^2}F_s^2 - F_1F_2 - F_2^2 &= \frac{1}{2}F_2^2 - \frac{q^2}{8m^2}F_2^2 - F_1F_2 - F_2^2,\\ &= -\frac{1}{2}\left[(F_2^2 + 2F_1F_2 + F_1^2 - F_1^2 + \frac{q^2}{4m^2}F_2^2\right],\\ &= -\frac{1}{2}\left[((F_1+F_2)^2 - \left(F_1^2 - \frac{q^2}{4m^2}F_2^2\right)\right].\end{aligned}$$

iii. $$(k'\cdot(p'+p))(k\cdot(p'+p)) - \frac{1}{2}(k'\cdot k)(p'+p)^2 = 4(Em)^2 + 2Emq^2 + \frac{q^4}{4} + q^2m^2 - \frac{q^4}{4}.$$

Combining these results, we see that the coefficient for the $(F_1+F_2)^2$ term will be

$$2(Em)^2 + Emq^2 + \frac{q^4}{4} + \frac{q^2}{2}m^2 - 2(Em)^2 - Emq^2 - \frac{q^2}{2}m^2 = \frac{q^4}{4},$$

which can be written,

$$\frac{q^4}{4} = \frac{q^2}{2}\frac{q^2}{2} = -\frac{2E^2m^2}{1+\frac{2E}{m}\sin^2\frac{\theta}{2}}\frac{q^2}{2m^2}.$$

Similarly, we will combine the results above to compute the coefficient for the $(F_1^2 - \frac{q^2}{4m^2}F_2^2)$ term.

$$\begin{aligned}2(Em)^2 + Emq^2 + \frac{q^2}{2}m^2 &= 2E^2m^2 - \frac{4E^3m\sin^2\frac{\theta}{2}}{1+\frac{2E}{m}\sin^2\frac{\theta}{2}} - \frac{2E^2m^2\sin^2\frac{\theta}{2}}{1+\frac{2E}{m}\sin^2\frac{\theta}{2}},\\ &= \frac{2E^2m^2 + 4E^3m\sin^2\frac{\theta}{2} - 4E^3m\sin^2\frac{\theta}{2} - 2m^2E^2\sin^2\frac{\theta}{2}}{1+\frac{2E}{m}\sin^2\frac{\theta}{2}},\\ &= 2E^2m^2\frac{1-\sin^2\frac{\theta}{2}}{1+\frac{2E}{m}\sin^2\frac{\theta}{2}} = \frac{2E^2m^2\cos^2\frac{\theta}{2}}{1+\frac{2E}{m}\sin^2\frac{\theta}{2}}.\end{aligned}$$

Therefore, combining all of these results, the total spin-average amplitude squared becomes

$$\overline{|\mathscr{M}|^2} = \frac{16e^4E^2m^2}{q^4\left(1+\frac{2E}{m}\sin^2\frac{\theta}{2}\right)}\left[\left(F_1^2 - \frac{q^2}{4m^2}F_2^2\right)\cos^2\frac{\theta}{2} - \frac{q^2}{2m^2}(F_1+F_2)^2\sin^2\frac{\theta}{2}\right].$$

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e) Let us compute the differential cross section, $\frac{d\sigma}{d\cos\theta}\big|_{\text{lab}}$. To do this, we will compute the cross section in most general terms. From elementary considerations, we calculated that

$$d\sigma = \frac{1}{2E_{\mathcal{A}}2E_{\mathcal{B}}|v_{\mathcal{A}}-v_{\mathcal{B}}|}\left(\prod_f \frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f}\right)\overline{|\mathscr{M}|^2}(2\pi)^4\delta^{(4)}\left(p_{\mathcal{A}}+p_{\mathcal{B}}-\sum p_f\right),$$
$$= \frac{1}{4mE}\overline{|\mathscr{M}|^2}\frac{d^3p'd^3k'}{(2\pi)^2}\frac{1}{4E'E'_p}\delta^{(4)}(p+k-p'-k').$$

We see that this is so because $E_{\mathcal{A}} = m$, $E_{\mathcal{B}} = E$, $|v_{\mathcal{A}} - v_{\mathcal{B}}| = 1$ and there are two final states.

Let us now integrate over $d\sigma$ to find its dependence on $\cos\theta$. During the derivation, we will make use of the fact that $E+m = E'_p+E'$ by energy conservation enforced by the dirac δ function. We will also call upon our results above to use the identities $E' = E + \frac{q^2}{2m}$ and $q^2 = \frac{4E^2\sin^2\frac{\theta}{2}}{1+\frac{2E}{m}\sin^2\frac{\theta}{2}}$. Notice the insertion of the Jacobian for the change of variables to integrate over the energy portion of the δ function in line 4. We will now proceed directly by first integrating over the p' part of the integral.

$$\sigma = \int d\sigma = \frac{1}{4mE}\overline{|\mathscr{M}|^2}\int\frac{d^3p'd^3k'}{(2\pi)^2}\frac{1}{4E'E'_p}\delta^{(4)}(p+k-p'-k'),$$
$$= \frac{1}{4mE}\overline{|\mathscr{M}|^2}\int\frac{d^3k'}{(2\pi)^2}\frac{1}{4E'E'_p}\delta^{(1)}\left(E'-E-m+\sqrt{m^2+E^2+E'^2-2EE'\cos\theta}\right),$$
$$= \frac{1}{4mE}\overline{|\mathscr{M}|^2}\int\frac{E'^2dEd\Omega}{(2\pi)^2}\frac{1}{4E'E'_p}\delta^{(1)}\left(E'-E-m+\sqrt{m^2+E^2+E'^2-2EE'\cos\theta}\right),$$
$$= \frac{1}{4mE}\overline{|\mathscr{M}|^2}\int\frac{d\Omega}{(2\pi)^2}\frac{E'}{4E'_p}\left(1+\frac{E'-E\cos\theta}{E'_p}\right)^{-1},$$
$$= \frac{1}{4mE}\overline{|\mathscr{M}|^2}\int\frac{d\Omega}{(2\pi)^2}\frac{E'}{4E'_p}\left(\frac{E'_p}{E'_p+E'-E\cos\theta}\right),$$
$$= \frac{1}{4mE}\overline{|\mathscr{M}|^2}\int\frac{d\cos\theta}{(2\pi)}\frac{E'}{4E'_p}\left(\frac{E'_p}{E'_p+E'-E\cos\theta}\right),$$
$$= \frac{1}{32\pi mE}\overline{|\mathscr{M}|^2}\int d\cos\theta\frac{E'}{m+E(1-\cos\theta)},$$
$$= \frac{1}{32\pi m^2E}\overline{|\mathscr{M}|^2}\int d\cos\theta\frac{E'}{1+\frac{2E}{m}\sin^2\frac{\theta}{2}},$$
$$= \frac{1}{32\pi m^2E}\overline{|\mathscr{M}|^2}\int d\cos\theta\frac{E+\frac{q^2}{2m}}{1+\frac{2E}{m}\sin^2\frac{\theta}{2}},$$
$$= \frac{1}{32\pi m^2E}\overline{|\mathscr{M}|^2}\int d\cos\theta\frac{E-\frac{2E^2\sin^2\frac{\theta}{2}}{m(1+\frac{2E}{m}\sin^2\frac{\theta}{2})}}{1+\frac{2E}{m}\sin^2\frac{\theta}{2}},$$
$$= \frac{1}{32\pi m^2E}\overline{|\mathscr{M}|^2}\int d\cos\theta\frac{E+\frac{2E^2}{m}\sin^2\frac{\theta}{2}-\frac{2E^2}{m}\sin^2\frac{\theta}{2}}{(1+\frac{2E}{m}\sin^2\frac{\theta}{2})^2},$$
$$= \frac{1}{32\pi m^2E}\overline{|\mathscr{M}|^2}\int d\cos\theta\frac{E}{(1+\frac{2E}{m}\sin^2\frac{\theta}{2})^2},$$
$$= \frac{1}{32\pi m^2(1+\frac{2E}{m}\sin^2\frac{\theta}{2})^2}\overline{|\mathscr{M}|^2}\int d\cos\theta,$$
$$= \frac{1}{32\pi m^2(1+\frac{2E}{m}\sin^2\frac{\theta}{2})^2}\overline{|\mathscr{M}|^2}\cos\theta,$$

$$\therefore\ \frac{d\sigma}{d\cos\theta}\bigg|_{\text{lab}} = \frac{1}{32\pi m^2\left(1+\frac{2E}{m}\sin^2\frac{\theta}{2}\right)^2}\overline{|\mathscr{M}|^2}.$$

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f) We will now derive the Rosenbluth formula. From our work above, we see that

$$\begin{aligned}
\left.\frac{d\sigma}{d\cos\theta}\right|_{\text{lab}} &= \frac{16e^4E^2m^2\left[\left(F_1^2 - \frac{q^2}{4m^2}F_2^2\right)\cos^2\frac{\theta}{2} - \frac{q^2}{2m^2}(F_1+F_2)^2\sin^2\frac{\theta}{2}\right]}{32\pi m^2\left(1+\frac{2E}{m}\sin^2\frac{\theta}{2}\right)^2 q^4\left(1+\frac{2E}{m}\sin^2\frac{\theta}{2}\right)}, \\
&= \frac{e^4E^2\left[\left(F_1^2 - \frac{q^2}{4m^2}F_2^2\right)\cos^2\frac{\theta}{2} - \frac{q^2}{2m^2}(F_1+F_2)^2\sin^2\frac{\theta}{2}\right]}{2\pi\frac{16E^4\sin^4\frac{\theta}{2}}{\left(1+\frac{2E}{m}\sin^2\frac{\theta}{2}\right)^2}\left(1+\frac{2E}{m}\sin^2\frac{\theta}{2}\right)^2\left(1+\frac{2E}{m}\sin^2\frac{\theta}{2}\right)}, \\
&= \frac{e^4\left[\left(F_1^2 - \frac{q^2}{4m^2}F_2^2\right)\cos^2\frac{\theta}{2} - \frac{q^2}{2m^2}(F_1+F_2)^2\sin^2\frac{\theta}{2}\right]}{32\pi E^2\sin^4\frac{\theta}{2}\left(1+\frac{2E}{m}\sin^2\frac{\theta}{2}\right)},
\end{aligned}$$

$$\therefore \left.\frac{d\sigma}{d\cos\theta}\right|_{\text{lab}} = \frac{\pi\alpha^2\left[\left(F_1^2 - \frac{q^2}{4m^2}F_2^2\right)\cos^2\frac{\theta}{2} - \frac{q^2}{2m^2}(F_1+F_2)^2\sin^2\frac{\theta}{2}\right]}{2E^2\sin^4\frac{\theta}{2}\left(1+\frac{2E}{m}\sin^2\frac{\theta}{2}\right)}.$$

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